

Energy conserving Anisotropic Anhysteretic Magnetic Modelling for Finite Element Analysis

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Abstract

To model ferromagnetic material in finite element analysis a correct description of the constitutive relationship (BH-law) must be found from measured data. This article proposes to use the energy density function as a centrepiece. Using this function, which turns out to be a convex function of the flux density, guarantees energy conservative modelling. The magnetic field strength can be seen as a derivative with respect to the flux density. Especially for anisotropic materials (from lamination and/or grain orientation) this method has advantages. Strictly speaking this method is only valid for anhysteretic and thermodynamically stable material.

1 Introduction

This essay discusses one aspect of numeric modelling of magnetic fields: anisotropic magnetic material properties. In practical electromagnetic applications anisotropic materials are common as laminated electric steel with and without grain orientation. In the design process finite element analysis (FEA) is widely used and correct modelling of the material properties is essential. The literature is wide spread, but considering saturation effects this essay contributes modelling that does not conflict with energy conservation.

Application with use of electric steels are transformers, motors, and generators. The main reason for lamination is that eddy currents in stacking direction are suppressed. This lamination also has the effect that the magnetic properties are anisotropic.

Two different types of electric steel are used: grain-oriented and non-grain-oriented steels. The non-grain-oriented sheets have magnetic properties that are isotropic in the sheet plane, whereas the non-grain-oriented steels have different magnetic properties in rolling direction and in transverse direction.

The discussion here is only applicable for perfect soft material, i.e. material which has a nonlinear BH-characteristic but has no hysteresis (meaning that the area of the hysteresis loop is empty). This is the requirement that there exists a one-to-one relationship between the vectorial flux density \mathbf{B} and the vectorial magnetic field \mathbf{H} . Models for hysteresis (Jiles-Atherton, Preisach, and other) are not considered here.

The centrepiece of this essay is the use of the magnetic energy as a function of the flux density as the centrepiece of the description of magnetic properties. The magnetic field strength is in this concept the gradient (with respect to the flux density) of the energy density. The energy density can be used to describe linear and nonlinear as well as isotropic and anisotropic material. It will be used to derive equations for lamination effects and as a base for the interpolation rule that extends measured data of grain-oriented steel to a full 3-D description.

Also the energy density function will be used to graph BH-characteristics by plotting lines of equal energy (iso-lines).

2 Maxwell-Equations

In order to clarify the notation, the relevant equations for magnetostatics are repeated. The flux density is described by a 3-vector \mathbf{B} which is divergence

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free (Gauß-law):

$$\nabla \cdot \mathbf{B} = 0.$$

The material properties are in the form of a function that maps \mathbf{B} to the magnetic field \mathbf{H} which is also a 3-vector. Formally:

$$\mathbf{H} = F(\mathbf{B}). \quad (1)$$

Ampere's-law relates the magnetic field to the electric current density vector \mathbf{j} , such that the curl of the magnetic field is equal to the current density:

$$\nabla \times \mathbf{H} = \mathbf{j}.$$

Many of the commonly used FEA software uses the vector potential \mathbf{A} as the unknown function, which is linked to the flux density by

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ 0 &= \nabla \cdot \mathbf{A}. \end{aligned}$$

The topic in this text is to discuss anisotropic materials which are described with a general nonlinear function F .

The differential reluctivity ν is a 3-by-3 matrix and defined as the partial derivative of the field strength with respect to the flux density. The components are

$$\nu_{ij} = \frac{\partial H_i}{\partial B_j}.$$

3 Literature review

Other authors often express the constitutive relation by employing the effective reluctivity matrix (or equivalently the effective permeability) in the form $\mathbf{H} = \nu_{eff} \mathbf{B}$, where ν_{eff} in general depends on the field quantities.

The case of lamination is often treated in the following way (eg. [11]): assuming steel has the material law

$$\mathbf{H} = \begin{pmatrix} \nu_{xx} & 0 & 0 \\ 0 & \nu_{yy} & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix} \mathbf{B}$$

and the lamination fill factors are f_1 for insulation and f_2 for steel. The BH-relationship for the composite is

$$\mathbf{H} = \begin{pmatrix} \frac{B_x}{f_1 \mu_0 + f_2 / \nu_{xx}} \\ \frac{B_y}{f_1 \mu_0 + f_2 / \nu_{yy}} \\ B_z \left(\frac{f_1}{\mu_0} + \nu_{zz} \right) \end{pmatrix} \mathbf{B}.$$

But this relationship is only correct if the reluctivity is constant. In reality the reluctivity must depend on the flux density and the reluctivity function of the underlying steel must be evaluated in the model using the flux density in the magnetic material (and not the macroscopic value). The reference [7] gives a correct model, where one additional equation must be solved. This agrees with the solution presented here, but is only valid for isotropic material.

When modelling the in-plane anisotropy of grain-oriented electric steel often only a limited number of sets of measure data (usually in rolling and transverse direction) is available. Then interpolation models must be found for intermediate directions. Usually a 2-D model is often presented in literature with the constitutive relation

$$\mathbf{H} = \begin{pmatrix} \nu_{xx} & \nu_{xy} \\ \nu_{yx} & \nu_{yy} \end{pmatrix} \mathbf{B}.$$

Such an effective reluctivity is not uniquely defined. There are two physically measurable degrees of freedom (H_x, H_y), but the symmetric reluctivity tensor contains three values ($\nu_{xx}, \nu_{xy} = \nu_{yx}, \nu_{yy}$). So when determining the reluctivity tensor a gauge has to be chosen, but no author actually discusses the implication of such a choice (eg. [3, 10, 2, 4]). Without going into the detailed equations the models all represent BH-relationships with non-symmetric differential reluctivity, and therefore violate energy conservation as detailed below.

In cases when measurements are available in the entire B_x, B_y -plane the interpolation can be done directly on this data ([5]).

The energy density function (or co-energy density function) is only used by few authors (eg. [12, 9]). The results presented here can be seen as an extension of these ideas so that the energy density forms the centrepiece of all treatment of BH-relationships and with special attention to anisotropy from lamination and grain-orientation.

4 General anisotropic BH-functions

In the introduction the BH-function (eq. 1) was considered to be a general vector function. Now the properties of this function are discussed.

Let γ be a path in the space of \mathbf{B} , i.e. a smooth function $[0, 1] \rightarrow R^3$. The energy density w related with this path is defined as

$$w(\gamma) = \int_{t=0}^1 \mathbf{H} \cdot \gamma' dt.$$

This energy is stored in the magnetic field at one material point. Energy conservation now means that w depends on the start and end points alone and not on the path. Using zero as the start point the energy density can be defined as a function of \mathbf{B} without referring to a particular path:

$$w(\mathbf{B}) = \int_{B'=0}^{\mathbf{B}} \mathbf{H} \cdot d\mathbf{B}.$$

The following statements follow from the existence of such an energy density function.

- The magnetic field is the gradient of the energy density

$$\mathbf{H} = \nabla_{\mathbf{B}} w$$

- Since the differential reluctivity tensor is the second derivative of w and is symmetric

$$\nu_{ij} = \frac{\partial H_i}{\partial B_j} = \frac{\partial^2 w}{\partial B_i \partial B_j}$$

must be fulfilled.

5 Convex energy function

For isotropic BH-characteristics it obvious that $|\mathbf{H}|$ is monotonously rising with $|\mathbf{B}|$. In consequence the derivative (the differential reluctivity) is positive. Ref. [8] show that for a thermodynamic stable material the differential permeability (and its inverse the differential reluctivity) must be positive. In an anisotropic case the reluctivity is a symmetric matrix and the requirement for positiveness is generalised as follows. A matrix M is called *positive definite* if for all vectors v the product $v \cdot Mv$ is always positive. It is therefore required that the differential reluctivity is such a positive definite matrix.

The requirement for a positive definite second derivative can be linked with the term of a *convex* function. A scalar function $f : R^n \rightarrow R$ is called

Magnetic energy density for vacuum

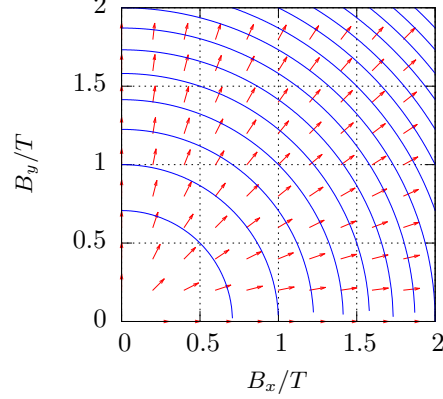


Figure 1: Contour lines of the energy density function for vacuum and direction of magnetic field strength

convex if for all $x, y \in R^n$ and all $t \in [0, 1]$ the inequality

$$f((1-t)x + ty) > (1-t)f(x) + tf(y)$$

holds. In words: the straight line connecting two points in the graph of the function lies *above* the function. The word *convex* can be explained, when looking at point sets like

$$U(Q) = \{x \in R^n | f(x) \leq Q\} \quad (2)$$

for $Q \in f(R^n)$. For convex functions these sets are convex in the ordinary sense. A set $U(Q)$ includes all points where the function takes values smaller or equal than Q . An appropriate way of displaying such functions is to look at iso-lines (where $f(x) = Q$), because these iso-lines enclose convex areas or volumes.

It can be demonstrated that for moderate additional assumption that convex functions have positive definite second derivatives and vice versa (see [1]).

The requirement for a positive definite reluctivity is also mentioned in [13]. It has the benefit that it guarantees a stable finite element analysis.

6 Examples

For simple cases the energy density functions are as follows. A natural way of graphing the energy

Energy density for anisotropic linear material

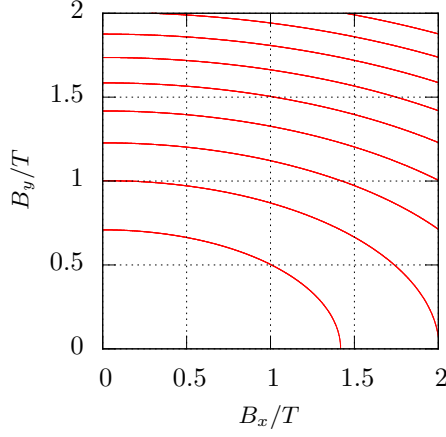


Figure 2: Contour lines of the energy density function for a linear anisotropic material.

Energy density for mild steel

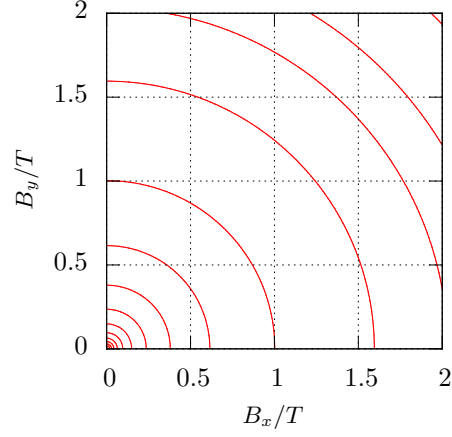


Figure 3: Contour lines of the energy density function for mild steel.

function in 2-D planes is to use contour lines of equal energy. These lines enclose convex areas if the energy function is valid.

- In vacuum:

$$w(\mathbf{B}) = \frac{\mathbf{B}^2}{2\mu_0} \quad (3)$$

with $\mu_0 = 4\pi 10^{-7}$ Vs/Am. As shown in fig. 1 lines of constant energy density are concentric circles around zero. This energy leads to the magnetic field strength:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}. \quad (4)$$

The mentioned figure also displays the direction of the field strength \mathbf{H} and shows that the field strength is perpendicular to isolines of the energy density.

- Anisotropic linear case:

$$w(\mathbf{B}) = \mathbf{B} \cdot \nu \mathbf{B},$$

where ν is the symmetric reluctivity matrix. The contour lines are concentric ellipses (fig. 2) and the BH-law is

$$\mathbf{H} = \nu \mathbf{B}.$$

- Nonlinear isotropic case: the energy density only depends on the modulus of the flux density

$$w(\mathbf{B}) = w_{iso}(|\mathbf{B}|). \quad (5)$$

The contour plot is similar to the plot for vacuum, only the spacing between the lines is different (fig. 3). The underlying BH-relationship can be measured and is displayed in fig. 5 The resulting BH-relationship is

$$\mathbf{H} = w'_{iso}(|B|) \frac{\mathbf{B}}{|\mathbf{B}|}. \quad (6)$$

The derivative w'_{iso} is exactly the result of measurements of BH-curves.

7 Lamination energy function

In principal, modelling a laminated core would require to resolve the individual layers of magnetic material and non-magnetic insulation in the FEA, which is due to the high number of sheets impractical. Therefore the method for homogenisation is used which models effective \mathbf{B} and \mathbf{H} fields instead. To derive the model the following is assumed:

- The coordinate system is such that the sheets are in xy -plane and stacked in z -direction
- Material 1 has the properties of vacuum (eq. 3)
- The magnetic properties for material 2 is described by a general energy density $w_2(\mathbf{B})$.
- The thickness of the sheets is so small that changes in the field from one sheet to its neighbouring sheet can be neglected, and in the same way from one insulation layer to the next (the fields in the sheet and the insulation, however, can be different)

The volume ratios material 1 and 2 are called f_1 and f_2 respectively.

From basic Maxwell-equations it is known that at interfaces the normal components of the flux density is continuous (Quantities with indexes 1 or 2 refer the respective materials, quantities without this additional index are homogenised values):

$$B_z := B_{1z} = B_{2z}$$

Homogenised flux density components are introduced for the non-continuous components by using a mixing rule, which is the natural choice from Maxwell-equations

$$\begin{aligned} B_x &:= f_1 B_{1x} + f_2 B_{2x} \\ B_y &:= f_1 B_{1y} + f_2 B_{2y}. \end{aligned}$$

Now, the homogenised energy density is a mixing of the energy density of the sublayers. Using B_{1x}, B_{1y} as the yet unknown flux densities in the insulation layer the energy density can be written as

$$\begin{aligned} w_{lam}^*(B_x, B_y, B_z, B_{1x}, B_{1y}) = & \frac{f_1}{2\mu_0} (B_{1x}^2 + B_{1y}^2 + B_z^2) \\ & + f_2 w_2 \left(\frac{1}{f_2} (B_x - f_1 B_{1x}), \frac{1}{f_2} (B_y - f_1 B_{1y}), B_z \right) \end{aligned}$$

which is the weighted sum of the energy densities in the two types of material. The flux density vector components are the values inside the respective material. Minimising the energy with respect to B_{1x}, B_{1y} gives

$$w_{lam}(\mathbf{B}) = \min_{B_{1x}, B_{1y}} w_{lam}^*(B_x, B_y, B_z, B_{1x}, B_{1y})$$

determines the unknowns B_{1x}, B_{1y} and fixes the energy density. Evaluating the partial derivatives and equation them to zero, results in the equations

$$\begin{aligned} \frac{B_{1x}}{\mu_0} &= F_x \left(\frac{1}{f_2} (B_x - f_1 B_{1x}), \frac{1}{f_2} (B_y - f_1 B_{1y}), B_z \right) \\ \frac{B_{1y}}{\mu_0} &= F_y \left(\frac{1}{f_2} (B_x - f_1 B_{1x}), \frac{1}{f_2} (B_y - f_1 B_{1y}), B_z \right), \end{aligned}$$

which constitute a system of nonlinear equations for B_{1x}, B_{1y} . Deriving with respect to B_x, B_y, B_z defines the magnetic field \mathbf{H} .

$$\mathbf{H} = \begin{pmatrix} 0 \\ 0 \\ \frac{f_1 B_z}{\mu_0} \end{pmatrix} + F \begin{pmatrix} \frac{1}{f_2} (B_x - f_1 B_{1x}) \\ \frac{1}{f_2} (B_y - f_1 B_{1y}) \\ B_z \end{pmatrix}$$

Putting the equations together the following set of equation describes a stack of the laminated electric steel:

$$\begin{pmatrix} H_x \\ H_y \\ \frac{1}{f_2} \left(H_z - \frac{f_1 B_z}{\mu_0} \right) \end{pmatrix} = F \begin{pmatrix} \frac{1}{f_2} (B_x - f_1 H_x \mu_0) \\ \frac{1}{f_2} (B_y - f_1 H_y \mu_0) \\ B_z \end{pmatrix}$$

Given the effective flux density the magnetic field can be computed by solving this system of nonlinear equations. This is now the material law for a stack of electric steel, if the underlying material law of the steel type is known (which will be discussed below). For general functions F these equation cannot be solved analytically. The system of equation has to be incorporated into a FEA software in a way that, every time the system enquires the value of the magnetic field for a given flux density, the system of equations is solved iteratively.

One step of approximation is as follows: if it is assumed that material 2 is strongly ferromagnetic then B_{1x} and B_{1y} can be neglected with respect to B_{2x} and B_{2y} respectively. The energy density is then approximated by

$$w_{lam}^{lin}(\mathbf{B}) = \frac{f_1 B_z^2}{2\mu_0} + f_2 w_2 \left(\frac{B_x}{f_2}, \frac{B_y}{f_2}, B_z \right).$$

This function can be evaluated without solving an additional nonlinear equation and the magnetic field derives as:

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} F_{2x} \left(\frac{B_x}{f_2}, \frac{B_y}{f_2}, B_z \right) \\ F_{2y} \left(\frac{B_x}{f_2}, \frac{B_y}{f_2}, B_z \right) \\ \frac{f_1 B_z}{\mu_0} + f_2 F_{2z} \left(\frac{B_x}{f_2}, \frac{B_y}{f_2}, B_z \right) \end{pmatrix}.$$

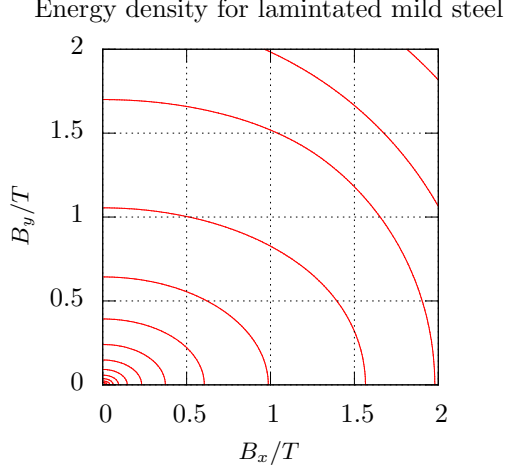


Figure 4: Contour lines of the energy density function for laminated mildsteel.

The advantage of this approximation for FEA is that it can be applied without solving additional equations. In practical cases, however, it must be verified that the accuracy of the approximation is sufficient.

In a first example this theory is applied to mild steel with an isotropic material behaviour in the manner of eq. 5. Figure 4 shows the lines of equal energy that results for a filling factor of $f_2 = 0.97$. Compared with fig. 3 - the bulk material - the lines appear to be squeezed towards the x -axis.

8 Grain-oriented electric steel

Now the method of the energy function shall be applied to model the magnetic characteristic of grain-oriented electric steel based on measurement. Typical measurements are along rolling direction and transverse direction like in fig. 5. These measurement are performed using the Epstein-frame by cutting sheets into stripes along the direction to be measured. In these measurements the flux density is aligned with the long edge of the stripes and the magnetic field strength component parallel to that edge is measured. The stripes are placed in x -direction, then the measurements gives as a func-

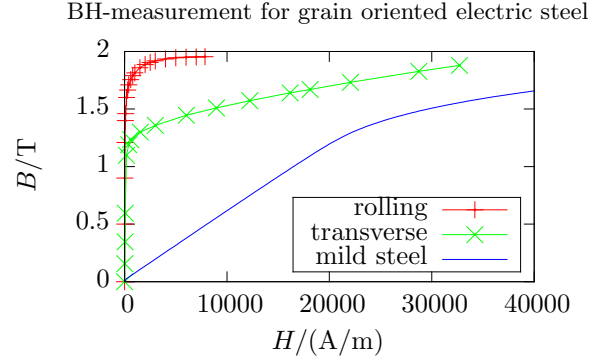


Figure 5: Measurement data for grain-oriented electric steel in rolling and in transverse direction compared with a measurement of mild steel.

tion $B \rightarrow H$ that is a part of the complete three-dimensional function:

$$H_{meas} = H_x(B_{meas}, 0).$$

In the same way a second measurement along y gives an additional view. From these two measurement an interpolation scheme is established for a two dimensional BH-characteristic. The functions from measurements will be called B_0 and B_{90} for the rolling direction and the transverse direction, respectively.

First the functions are integrated:

$$\begin{aligned} w_0(B) &= \int_0^B H_0(B') dB' \\ w_{90}(B) &= \int_0^B H_{90}(B') dB'. \end{aligned}$$

These functions are the basis for an interpolation rule for the energy density function $w : B_x, B_y \rightarrow w(B_x, B_y)$. The main point is that the lines of equal energy are constructed for all values of w which sufficiently defines the function. For a given value w the inverse functions of w_0, w_{90} are used to find B_0, B_{90} such that

$$w = w_0(B_0) = w_{90}(B_{90}).$$

The iso-line is defined to be an ellipse:

$$\left(\frac{B_x}{B_0(w)} \right)^2 + \left(\frac{B_y}{B_{90}(w)} \right)^2 = 1 \quad (7)$$

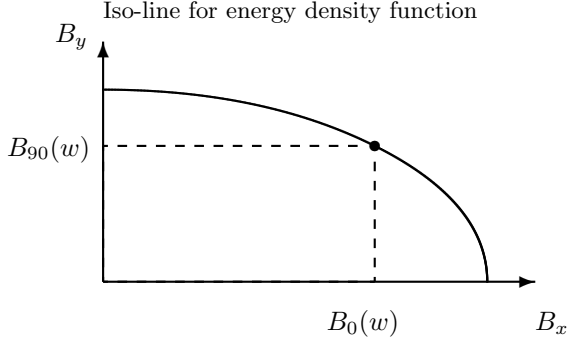


Figure 6: The iso-line for energy density w is constructed of be an ellipse that meets measured results at the axes.

Apparently the enclosed area is convex which is essential because a convex interpolation function must be found. In a practical implementation the flux density is given and the energy density must be found. A system of two equations has to be solved in this case. The two equations are the ellipse equation and the requirement that the half axes of the ellipse correspond to the same energy. To be solved for B_0, B_{90} :

$$\left(\frac{B_x}{B_0}\right)^2 + \left(\frac{B_y}{B_{90}}\right)^2 = 1 \quad (8)$$

$$w_0(B_0) = w_{90}(B_{90}). \quad (9)$$

From the ellipse equation the equation determining the magnetic field strength are derived by taking the partial derivatives with respect to the flux density components:

$$\begin{pmatrix} H_x \\ H_y \end{pmatrix} = \frac{1}{\frac{B_x^2}{B_0^3 H_0} + \frac{B_y^2}{B_{90}^3 H_{90}}} \begin{pmatrix} B_x / B_0^2 \\ B_y / B_{90}^2 \end{pmatrix} \quad (10)$$

Here only the two-dimensional case is shown, the extension to three dimensions is obvious and leads to a system of three equations.

9 Application example

The interpolation scheme shall be applied to the measured BH-curves shown in fig. 5. The measurements are limited to below 2 T, but when applications in the saturation regime are targeted,

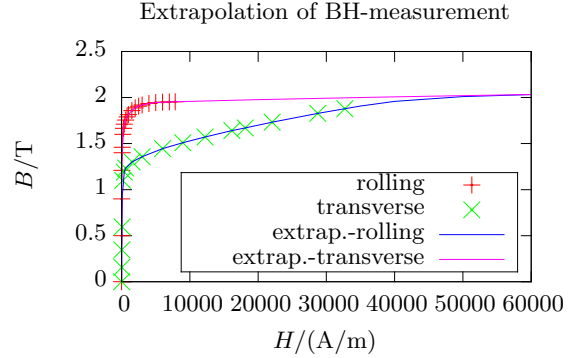


Figure 7: Measurement data for grain-oriented electric steel in rolling and in transverse direction and extrapolated data to capture saturation.

the measured data must be extrapolated. For the case of the rolling direction saturation is practically reached and the curve can be extended using the permeability of one:

$$B_{highfield} = B_{sat} + \frac{H}{\mu_0}.$$

The case for the transverse direction is more complicated. Reference [6]) states that in any direction the saturation flux density is the same. Therefore a continuation is constructed that approaches the same saturation equation. This is depicted in fig. 7.

The application of the interpolation scheme in the sheet plane give the iso-line plot shown in fig. 8. The ellipses are clearly anisotropic but tend to become closer to circles at higher flux densities of ≈ 3 T when saturation is reached and the material becomes isotropic.

The next step is to combine the result of the grain-oriented material with the lamination effects. Here measurements for the BH-characteristic in direction perpendicular to the sheet are needed in order to apply the interpolation scheme in three dimensions. There are no such measurements publicly available. The interpolation rule with two extreme cases is demonstrated. One uses the BH-curve of the rolling direction also for the out-of-plane direction. The other case uses the measurement in transverse direction as BH-curve in the perpendicular direction. The two results for the energy density are both displayed in fig. 9. These iso-lines show are cut in the plane of rolling direction and stacking di-

Isolines of energy for grain-oriented steel

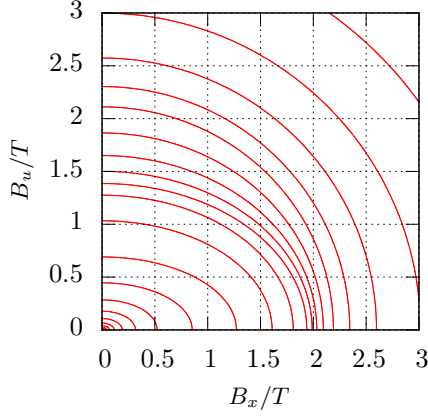


Figure 8: Iso-lines of magnetic energy for grain-oriented electric steel based on measurements along rolling direction(x) and transverse direction(y); other points according presented interpolation scheme.

rection of the three-dimensional picture. It can be seen that the difference between both cases is small. For points with dominating *rolling* component the two models naturally give same results. With dominating *stacking* component the filling factor is determining the result, hence both models give the similar results.

10 Conclusion

This article proposes the use of the energy density function as the appropriate quantity to describe magnetic constitutive relations. Energy consumption is guaranteed. The dependence of the field strength on the flux density is found by derivation. The differential reluctivity is automatically symmetric.

For a thermodynamically stable material the energy density is a convex function of the flux density. This leads to a positive definite differential reluctivity and the finite-element-method is stable.

The method is in particular suitable for anisotropic materials. The BH-law for anisotropy from

Isolines of energy for grain-oriented steel

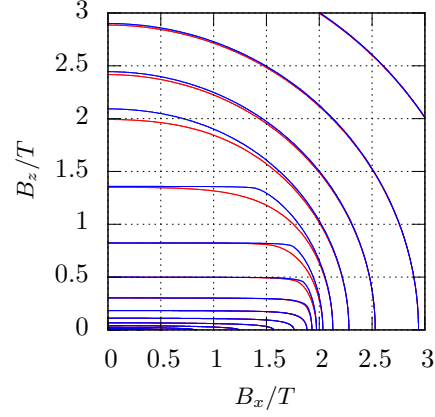


Figure 9: Iso-lines if energy density of grain oriented electric steel with lamination effect. Model 1 (red line) uses the BH-curve of the transverse direction in perpendicular direction; model 2 (blue line) uses the measurement from rolling direction.

lamination is demonstrated. For grain oriented steel an anisotropic material law is derived from two measurements (along rolling direction and transverse directions). The interpolation for arbitrary directions is done using the energy density and by building isosurfaces for this function by using measured data as starting points. This BH-law is used as input to the previously found lamination law.

If measurements are available in intermediate directions the data can included in the isosurface construction.

It must be repeated that the presented method is only applicable for anhysteretic material because only then the energy density is a function of one variable (the flux density) alone. Furthermore the measurements must be performed such that the material is always in a thermodynamic stable state, on then the convexity condition is valid.

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